

Practice Second Midterm Exam IV

This exam is closed-book and closed-computer. You may have a double-sided, 8.5" × 11" sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 36 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

Question	Points
(1) Binary Relations	/ 10
(2) Functions	/ 8
(3) Graph Theory	/ 8
(4) Induction	/ 10
	/ 36

Problem One: Binary Relations**(10 Points)***(Midterm Exam, Winter 2016)*

On Problem Set Three, you explored binary relations and formal definitions specified in first-order logic. This question explores a way of forming new binary relations and asks you to prove properties about them. In the course of doing so, we hope you'll be able to demonstrate what you've learned about binary relations and proving results about terms defined in first-order logic.

Let's suppose that we have an arbitrary strict order R over a set A . We can use R to define a new binary relation \tilde{R} over the set $\wp(A)$ as follows:

$$X\tilde{R}Y \quad \text{if} \quad Y \neq \emptyset \wedge \forall x \in X. \forall y \in Y. xRy.$$

This relation \tilde{R} is called the *lift of R to $\wp(A)$* .

Prove that if R is an arbitrary strict order over a set A , then \tilde{R} is a strict order over the set $\wp(A)$.

(Extra space for your answer to Problem One, if you need it.)

Problem Two: Functions**(8 Points)***(Midterm Exam, Fall 2015)*

On Problem Set Three, we introduced **right inverses** and asked you to prove some properties about them. In this question, we're going to ask you to revisit right inverses and prove another property about them. In the course of doing so, you'll get a chance to demonstrate what you've learned about injections, surjections, and bijections.

Let's begin by refreshing some of the terminology from Problem Set Three. If $f : A \rightarrow B$ is a function, then we say that a function $g : B \rightarrow A$ is a **right inverse of f** if $f(g(b)) = b$ for every $b \in B$. If f is a function that has a right inverse, then we say that f is **right-invertible**. On Problem Set Three, you proved that all right-invertible functions are surjective.

Let $f : A \rightarrow B$ be an arbitrary right-invertible function and let $g : B \rightarrow A$ be one of its right inverses. Prove that if g is surjective, then f is a bijection.

(Extra space for your answer to Problem Two, if you need it.)

Problem Three: Graph Theory**(8 Points)***(Midterm Exam, Fall 2015)*

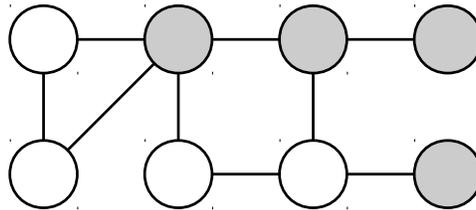
If you'll recall from Problem Set Four, an **independent set** in a graph $G = (V, E)$ is a set $I \subseteq V$ with the following property:

$$\forall u \in I. \forall v \in I. \{u, v\} \notin E.$$

Let's begin with a new definition. A **dominating set** in G is a set $D \subseteq V$ with the following property:

$$\forall v \in V. (v \in D \vee \exists u \in D. \{u, v\} \in E)$$

For example, in the graph given below, the nodes in gray form a dominating set:



This question explores the interplay between independent sets and dominating sets.

- i. **(4 Points)** Let $G = (V, E)$ be a graph with the following property: every node in G is adjacent to at least one other node in G . Prove that if I is an independent set in G , then $V - I$ is a dominating set in G . (Notice that we're asking you to show that $V - I$ is a dominating set, not that I is a dominating set.)

As a refresher, a **dominating set** in G is a set $D \subseteq V$ with the following property:

$$\forall v \in V. (v \in D \vee \exists u \in D. \{u, v\} \in E)$$

Now, let's introduce a new definition. We'll say that an independent set I in a graph G is a **maximal independent set** in G if there is no independent set I' in G where $I \subsetneq I'$. (Here, $I \subsetneq I'$ denotes that I is a strict subset of I'). Intuitively, a maximal independent set is one that can't be enlarged to form an even bigger independent set.

- ii. **(4 Points)** Let $G = (V, E)$ be any undirected graph. Prove that if I is a maximal independent set in G , then I is also a dominating set in G .

Problem Four: Induction**(10 Points)***(Midterm Exam, Fall 2016)*

This problem explores mathematical induction as applied to binary relations. We hope that it lets you show what you've learned about how to set up and structure a proof by induction while calling back to terms defined in first-order logic.

Given a binary relation R over a set A and a natural number $n \geq 1$, we can define a new binary relation over A called the ***n*th power of R** , denoted R^n . This relation is defined inductively as follows:

$$\begin{aligned}xR^1y & \text{ if } xRy \\xR^{n+1}y & \text{ if } \exists z \in A. (xRz \wedge zR^n y).\end{aligned}$$

(Note that R^n is only defined for $n \geq 1$.)

Let R be an arbitrary ***transitive*** relation over a set A . Prove, by induction, that the following statement is true for all natural numbers $n \geq 1$:

For any $x, y \in A$, if $xR^n y$, then xRy .

(Extra space for your answer to Problem Four, if you need it.)